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AN INVESTIGATION OF ELASTIC FIELD CREATED BY RANDOMLY DISTRIBUTED INCLUSIONS

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Abstract—In this paper, the elastic field created by randomly distributed inclusions is studied. The inclusions are considered to be randomly distributed in the material, and have random orientation and size. The random point field model is proposed to describe the randomness of inclusion position, orientation and size. As a special case, when phase transformation inclusions are uniformly distributed in the material, and have non-random orientation, the theory gives the same result as Mori and Tanaka (1973, *Acta Metallurgica* 21, 571). The elastic field created by randomly distributed dislocation loops is also considered in some detail, and it is found that the continuum theory of dislocation loops is applicable only when the size of the dislocation loop becomes infinitesimal.

INTRODUCTION

An inclusion is defined as a sub-domain Ω in domain D, where eigenstrain is given in Ω, and is zero in $D-\Omega$. It may represent a phase transformation particle, a dislocation loop, etc., which are the sources of internal stress. The existence of inclusions can greatly influence the properties of the material. Therefore, it is important to investigate the elastic field created by inclusions. Seldom is only a single inclusion formed in a material, and it is natural that the random distribution of inclusions enters in consideration.

Mori and Tanaka (1973) have calculated the average internal stress in the matrix of a material containing inclusions with transformation strain. They assumed that the average stress $\langle\sigma_{ij}^M\rangle$ in the matrix is constant, and derived the stress field $\langle\sigma_{ij}^I\rangle$ within an inclusion using Eshelby's famous solution (1957). By the aid of the ergodic theorem, i.e.

$$v_f\langle\sigma_{ij}^I\rangle+(1-v_f)\langle\sigma_{ij}^M\rangle = 0, \tag{1}$$

where v_f is the volume fraction of inclusion, the average stress in the matrix was calculated. Tandon and Weng (1986) have further developed Mori and Tanaka's method to obtain the stress field in a material containing inhomogeneities with different elastic modulus from the matrix. These approaches are simple, and are intuitive methods with some understanding of randomness. They are applicable only for uniform distribution of inclusions with non-random orientation.

The present investigation attempts to develop a statistical theory which can be applicable to a more general random inclusion problem. The random point field model is proposed to describe the randomness of inclusion position, orientation and size. As a special case, when phase transformation inclusions are uniformly distributed in the material, and have non-random orientation, the theory gives the same result as Mori and Tanaka. The elastic field created by randomly distributed dislocation loops is also considered in some detail, and it is found that the continuum theory of dislocation loops is applicable only when the size of dislocation loop becomes infinitesimal.

BASIC THEORY

The equation of classical elasticity can be written in the form

$$-\partial_j C_{ijkl}\,\partial_k u_l = q_i \tag{2}$$

where u_l is the displacement component which is zero for the stress-free state. If some regions of material are subjected to such non-elastic deformations as thermal expansion, phase transformation, etc., which are called eigen-displacements, the total displacement component u_l^S is,

$$u_l^S = u_l + u_l^{\mathrm{T}}. \tag{3}$$

By substituting eqn (3) into eqn (2), the equation becomes,

$$-\partial_j C_{ijkl}\,\partial_k(u_l^S - u_l^{\mathrm{T}}) = q_i \tag{4}$$

or,

$$-\partial_j C_{ijkl}\,\partial_k u_l^S = q_i - \partial_i C_{ijkl}\,\partial_k u_l^{\mathrm{T}}. \tag{5}$$

If the Green's function for an unbounded medium is defined as $G(\mathbf{x}-\mathbf{x}')$,

$$\partial_j C_{ijkl}\,\partial_k G_{lm}(\mathbf{x}-\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')\delta_{im} \tag{6}$$

we obtain

$$u_p^S = u_p^0 - \int_v G_{pi}(\mathbf{x}-\mathbf{x}')\,\partial_j C_{ijkl}\,\partial_k u_l^{\mathrm{T}}\,\mathrm{d}v\,(\mathbf{x}') \tag{7}$$

where v is the region taken by inclusions, and u_p^0 is the displacement which would be presented in the homogeneous medium under the action of an external stress field.

Let us apply the operator def ($\varepsilon = \mathrm{def}\,u$) to both sides of eqn (7), taking into account the symmetry of C_{ijkl}, we obtain the equation for the strain,

$$\varepsilon_{ij} = \varepsilon_{ij}^0 + \int_v K_{ijkl}(\mathbf{x}-\mathbf{x}')C_{klpq}\varepsilon_{pq}^{\mathrm{T}}\,\mathrm{d}v\,(\mathbf{x}') \tag{8}$$

where ($\varepsilon_0 = \mathrm{def}\,u_0$) is an external strain field, ε_{pq}^T is the eigenstrain, and

$$\begin{aligned} K_{ijkl} &= -(\partial_i\,\partial_l G_{jk})_{(ik)(jl)} \\ &= -\tfrac{1}{4}(\partial_i\,\partial_l G_{jk} + \partial_j\,\partial_l G_{ik} + \partial_i\,\partial_k G_{jl} + \partial_j\,\partial_k G_{il}). \end{aligned} \tag{9}$$

The equation for stress follows from eqn (8), where $\sigma_{ij}^0 = C_{ijkl}\varepsilon_{kl}^0$,

$$\sigma_{ij} = \sigma_{ij}^0 - \int_v S_{ijkl}(\mathbf{x}-\mathbf{x}')\varepsilon_{kl}^{\mathrm{T}}\,\mathrm{d}v\,(\mathbf{x}') \tag{10}$$

where

$$S_{ijkl} = C_{ijkl}\delta(\mathbf{x}'-\mathbf{x}) - C_{ijpq}K_{pqmn}C_{mnkl}.$$

The basic equations for eigenstrain problems are derived as eqns (8) and (10). In what follows, we consider that the eigenstrain occurs in located regions. In such a case, the eigenstrain can be represented in the form

$$\varepsilon_{pq}^{T}(\mathbf{x}) = \sum_{a=1}^{Nv} \varepsilon_{pq}^{*} V_a(\mathbf{x}) \tag{11}$$

where $V_a(x)$ is the characteristic function of the region occupied by the a-th inclusion. The tensor ε_{pq}^{*} is a constant random variable within the inclusions, and Nv is the number of inclusions in the volume v.

$$V_a(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in V_a \\ 0 & \mathbf{x} \notin V_a \end{cases} \tag{12}$$

where V_a is the region of the a-th inclusion. By substituting eqn (11) into eqns (8) and (10), we derive that

$$\varepsilon_{ij} = \varepsilon_{ij}^{0} + \sum_{a=1}^{Nv} \int_{v} K_{ijkl}(\mathbf{x}-\mathbf{x}') C_{klpq} \varepsilon_{pq}^{*} V_a(\mathbf{x}')\, \mathrm{d}v(\mathbf{x}') \tag{13}$$

$$\sigma_{ij} = \sigma_{ij}^{0} - \sum_{a=1}^{Nv} \int_{v} S_{ijkl}(\mathbf{x}-\mathbf{x}') \varepsilon_{kl}^{*} V_a(\mathbf{x}')\, \mathrm{d}v(\mathbf{x}'). \tag{14}$$

Since the location, size and orientation of inclusions are all random variables, the stress and strain determined by eqns (13) and (14) are random field variables. In order to derive their statistical characteristics, we propose the Random Point Field Model as follows.

Assumptions

(1) In volume V, the number of inclusions obeys the Poisson distribution with parameter λ, i.e. for $n = 0, 1, 2, \ldots$

$$P_r[N_v = n] = (n!)^{-1} \left[\int_v \lambda(\mathbf{r})\, \mathrm{d}v(\mathbf{r}) \right]^n \exp\left[- \int_v \lambda(\mathbf{r})\, \mathrm{d}v(\mathbf{r}) \right] \tag{15}$$

where $P_r[N_v = n]$ is the probability of $N_v = n$ and λ is the mean value of inclusion number in unit volume. (2) $\{N_v;\, v \subset V\}$ has independent increments in regions with a restriction for the intersection of them, i.e. for $v = v_1, v_2, \ldots, v_k$

$$P_r[N_{v_1} = n_1, N_{v_2} = n_2, \ldots, N_{v_k} = n_k] = \prod_{i=1}^{k} P_r[N_{v_i} = n_i]. \tag{16}$$

According to eqns (13) and (14), the perturbation term in strain and stress field caused by random inclusions can be represented in the general form:

$$A_{ij}(\mathbf{x}) = \sum_{a=1}^{Nv} A_{ij}^{(a)}(\mathbf{x} - \mathbf{r}_a; \Phi_a) \tag{17}$$

where $\mathbf{r}_a$ is the center location of the a-th inclusion, and Φ_a stands for the orientation and size of the a-th inclusion. The characteristic function of random field variable A_{ij} is defined as

$$M_A = E[\exp(I a_{ij} A_{ij})] \tag{18}$$

where a_{ij} is a constant tensor and I is the imaginary unit.

By substituting eqn (17) into eqn (18), the detailed expression for M_A can be derived as follows

$$\tilde{M}_A = E\left[\exp\left(Ia_{ij}\sum_{a=1}^{N_v} A_{ij}^{(a)}(\mathbf{x}-\mathbf{r}_a;\Phi_a)\right)\right]. \tag{19}$$

By using the properties of conditional expectation, eqn (19) can be represented in the form

$$M_A = P_r[N_v = 0] + \sum_{n=1}^{\infty} P_r[N_v = n]E\left\{\exp\left[I\sum_{a=1}^{n} a_{ij}A_{ij}^{(a)}\right]\Bigg|_{N_v=n}\right\} \tag{20}$$

where $E[\ ]$ denotes the mean for the orientation of an inclusion. According to the assumptions, it can be obtained from (see Wang, 1988)

$$E\left\{\exp\left[I\sum_{a=1}^{n} a_{ij}A_{ij}^{(a)}(\mathbf{x}-\mathbf{r}_a;\Phi_a)\right]\Bigg|_{N_v=n}\right\}$$
$$= \left\{\left(\int_v \lambda\,\mathrm{d}v\right)^{-1}\int_v \lambda E\left\{\exp\left[Ia_{ij}A_{ij}^{(a)}(\mathbf{x}-\mathbf{r}_a;\Phi_a)\right]\right\}\mathrm{d}v\,(\mathbf{r}_a)\right\}^n. \tag{21}$$

By substituting eqns (15) and (21) into eqn (20), the detailed expression for M_A can be represented in the form

$$M_A = \exp\left\{\int_v \lambda E[\exp\,(Ia_{ij}A_{ij}^{(a)}) - 1]\,\mathrm{d}v\,(\mathbf{r}_a)\right\}. \tag{22}$$

Given y_n is the cumulant of the n-th order and by using the properties of the characteristic function, we know;

$$I^n y_n = (\partial^n M_A/\partial a^n) \qquad (a_{ij} = 0). \tag{23}$$

So, the mean value is

$$\langle A_{ij}\rangle = \int_v \lambda E[A_{ij}^{(a)}(\mathbf{x}-\mathbf{r}_a;\Phi_a)]\,\mathrm{d}v\,(\mathbf{r}_a). \tag{24}$$

For the same reason, the correlation function can be derived easily.

According to eqn (24), the mean value of the strain field and stress field at point x can be represented in the form

$$\langle \varepsilon_{ij}\rangle = \varepsilon_{ij}^0 + \int_v \lambda(\mathbf{r}_a)E\left[\int_{v_a} K_{ijkl}(\mathbf{x}-\mathbf{x}')C_{klpq}\varepsilon_{pq}^*\,\mathrm{d}v(\mathbf{x}')\right]\mathrm{d}v\,(\mathbf{r}_a) \tag{25}$$

$$\langle \sigma_{ij}\rangle = \sigma_{ij}^0 - \int_v \lambda(\mathbf{r}_a)E\left[\int_{v_a} S_{ijkl}(\mathbf{x}-\mathbf{x}')\varepsilon_{kl}^*\,\mathrm{d}v\,(\mathbf{x}')\right]\mathrm{d}v(\mathbf{r}_a) \tag{26}$$

where v_a is the region of the ath inclusion with its center at $\mathbf{r}_a$. If it is assumed that the center locations of inclusions form a Homogeneous Poisson Field, i.e. $\lambda = m$ (const.), eqns (25) and (26) become

$$\langle \varepsilon_{ij} \rangle = \varepsilon_{ij}^0 + m \int_v E\left[\int_{v_0} K_{ijkl}(\mathbf{x} - \Delta - \mathbf{r}_a) C_{klpq} \varepsilon_{pq}^* \, dv(\Delta) \right] dv(\mathbf{r}_a) \tag{27}$$

$$\langle \sigma_{ij} \rangle = \sigma_{ij}^0 - m \int_v E\left[\int_{v_0} S_{ijkl}(\mathbf{x} - \Delta - \mathbf{r}_a) \varepsilon_{kl}^* \, dv(\Delta) \right] dv(\mathbf{r}_a) \tag{28}$$

where v_0 is the region of the inclusion with center at zero, and $\mathbf{x}' = \Delta + \mathbf{r}_a$.

(1) *The average field for an unbounded medium*

According to Kunin's discussion (1983), we know that:

$$\int_\infty K_{ijkl}(\mathbf{x} - \Delta - \mathbf{r}_a) \, dv(\Delta) = B_{ijkl}$$

$$\int_\infty S_{ijkl}(\mathbf{x} - \Delta - \mathbf{r}_a) \, dv(\Delta) = 0 \tag{29}$$

where B_{ijkl} is the elastic compliance tensor. As a consequence of eqn (29), it is obtained that

$$\langle \varepsilon_{ij} \rangle = \varepsilon_{ij}^0 + v_f \langle \varepsilon_{ij}^* \rangle$$

$$\langle \sigma_{ij} \rangle = \sigma_{ij}^0 \tag{30}$$

where v_f is the volume fraction of inclusions. Equation (30) is just the result which can be obtained according to the ergodic theorem.

(2) *The mean value of field variables within an inclusion*

In this case, we can assume that there is an inclusion at point x, that is,

$$\lambda(\mathbf{r}_a) = \begin{cases} \delta(\mathbf{r}_a - \mathbf{x}) & \mathbf{r}_a \in V_x \\ m & \mathbf{r}_a \notin V_x \end{cases} \tag{31}$$

where V_x is the region of the inclusion with center at x. If $\mathbf{r}_a \in V_x$, and the inclusions are ellipsoidal inclusions, the following expressions can be obtained (see Kunin, 1983)

$$\int_{v_0} K_{ijkl}(\mathbf{x} - \Delta - \mathbf{r}_a) \, dv(\Delta) = A_{ijkl} \tag{32}$$

$$\int_{v_0} S_{ijkl}(\mathbf{x} - \Delta - \mathbf{r}_a) \, dv(\Delta) = D_{ijkl} \tag{33}$$

where the components of tensor A and D are shown in the Appendix. By substituting eqns (32) and (33) into eqns (25) and (26), the mean values of strain and stress within an inclusion are obtained as

$$\langle \varepsilon_{ij}^I \rangle = \varepsilon_{ij}^0 + \langle \varepsilon_{ij}^* \rangle + (1 - v_f) \langle A_{ijkl} C_{klpq} \varepsilon_{pq}^* \rangle \tag{34}$$

$$\langle \sigma_{ij}^I \rangle = \sigma_{ij}^0 - (1 - v_f) \langle D_{ijkl} \varepsilon_{kl}^* \rangle \tag{35}$$

where v_f is the volume fraction of inclusions.

(3) *The mean value of field variables in the matrix*

In this case, we can assume that there is no inclusion at point x, i.e.

$$\lambda(\mathbf{r}_a) = \begin{cases} 0 & \mathbf{r}_a \in V_x \\ m & \mathbf{r}_a \notin V_x. \end{cases} \tag{36}$$

By substituting eqns (36), (32) and (33) into eqns (25) and (26), the mean values of strain and stress field in the matrix are obtained as

$$\langle \varepsilon_{ij}^M \rangle = \varepsilon_{ij}^0 + v_f \langle \varepsilon_{ij}^* \rangle - v_f \langle A_{ijkl} C_{klpq} \varepsilon_{pq}^* \rangle \tag{37}$$

$$\langle \sigma_{ij}^M \rangle = \sigma_{ij}^0 + v_f \langle D_{ijkl} \varepsilon_{kl}^* \rangle \tag{38}$$

where $A_{ijkl} = s_{ijpq} B_{pqkl}$ and $D = C - C:A:C$, s_{ijpq} is the Eshelby's tensor. The stress determined by eqn (38) is same as Mori and Tanaka's result (1973) if the randomness of inclusion orientation does not enter in consideration.

CALCULATION FOR RANDOMLY DISTRIBUTED DISLOCATION LOOPS

According to Mura (1982), the eigenstrain ε_{ij}^* which is caused by the slip b_i of plane s whose normal vector is n_i can be represented in the form

$$\varepsilon_{ij}^*(\mathbf{x}) = -\tfrac{1}{2}(b_i n_j + b_j n_i)\delta(s - \mathbf{x}) \tag{39}$$

where $\delta(s - \mathbf{x})$ is the one-dimensional Dirac Delta Function in the normal direction of s, i.e.

$$\int \delta(s - \mathbf{x})\, \mathrm{d}v(\mathbf{x}) = \int \mathrm{d}s. \tag{40}$$

(1) *Comparison with the continuum theory of dislocation loop*

The elastic strain created by a single dislocation loop is obtained from Mura (1982) as

$$\varepsilon_{ij} = \int_\Omega K_{ijkl}(\mathbf{x} - \mathbf{x}') C_{klmn} b_m n_n \, \mathrm{d}s(\mathbf{x}') \tag{41}$$

where Ω is the slip plane. Kroupa (1962) has introduced the dislocation loop density tensor as

$$\beta_{mn}\, \mathrm{d}v(\mathbf{x}') = b_m n_n \, \mathrm{d}s(\mathbf{x}'). \tag{42}$$

Substitution of eqn (42) into eqn (41) gives

$$\varepsilon_{ij} = \int_v K_{ijkl}(\mathbf{x} - \mathbf{x}') C_{klmn} \beta_{mn} \, \mathrm{d}v(\mathbf{x}') \tag{43}$$

where v is the region occupied by continuously distributed dislocation loops. When determining the tensor β_{mn} in practice, Kroupa has adopted the following procedure: from the body we take a volume Δv, in which there is a larger number of loops, and divide the loops into N groups; in the general Kth group are loops with the same Burgers vector $b_m^{(k)}$, the same normal $n_n^{(k)}$ and the same area $A^{(k)}$, of which there are $M^{(k)}$. Then

$$\beta_{mn} = \sum_{k=1}^{N} M^{(k)} A^{(k)} n_n^{(k)} b_m^{(k)} / \Delta V. \tag{44}$$

If we assume that $A^{(k)} n_n^{(k)} b_m^{(k)}$ is the same for every group, one obtains

$$\beta_{mn} = A b_m n_n \sum_{k=1}^{N} M^{(k)} / \Delta V = \lambda A b_m n_n \tag{45}$$

where λ is the number of dislocation loop in unit volume. Substitution of eqn (45) into eqn (43) gives

$$\varepsilon_{ij} = A \int_v \lambda K_{ijkl}(\mathbf{x}-\mathbf{x}') C_{klmn} b_m n_n \, \mathrm{d}v\,(\mathbf{x}'). \tag{46}$$

Equation (46) is the result obtained from the continuum theory of dislocation loops. Whereas, according to eqns (25) and (39), we know

$$\varepsilon_{ij} = \int_v \lambda \, \mathrm{d}v\,(\mathbf{r}) \int_{\Omega(r)} K_{ijkl}(\mathbf{x}-\mathbf{x}') C_{klmn} b_m n_n \, \mathrm{d}s\,(\mathbf{x}) \tag{47}$$

where $\Omega(r)$ is the area of dislocation loop with center at r. For infinitesimal dislocation loop, one can approximately obtain

$$\int_{\Omega(r)} K_{ijkl}(\mathbf{x}-\mathbf{x}') C_{klmn} b_m n_n \, \mathrm{d}s\,(\mathbf{x}') \doteq A K_{ijkl}(\mathbf{x}-\mathbf{r}) C_{klmn} b_m n_n. \tag{48}$$

By substituting eqn (48) into eqn (47), the same result as eqn (46) will be derived. This may mean that the eqn (43) is applicable only for infinitesimal dislocation loop distribution.

By substituting eqn (39) into eqns (25) and (26), the following results can be derived.

(2) *The average field for an unbounded medium*

$$\langle \varepsilon_{ij} \rangle = \varepsilon_{ij}^0 - mA \langle \tfrac{1}{2}(n_i b_j + n_j b_i) \rangle \tag{49}$$

$$\langle \sigma_{ij} \rangle = \sigma_{ij}^0 \tag{50}$$

where A is the mean area of a dislocation loop.

(3) *The mean value of field variables within a dislocation loop*

$$\langle \varepsilon_{ij}^I \rangle = \varepsilon_{ij}^0 - mA \langle \tfrac{1}{2}(n_i b_j + n_j b_i) \rangle - \tfrac{1}{2}(1-mA) \langle \bar{A}_{ijkl} C_{klpq}(n_p b_q + n_q b_p) \rangle \tag{51}$$

$$\langle \sigma_{ij}^I \rangle = \sigma_{ij}^0 + \tfrac{1}{2}(1-mA) \langle \bar{D}_{ijkl}(n_k b_l + n_l b_k) \rangle \tag{52}$$

where the components of tensor A and D are shown in the Appendix.

(4) *The mean value of field variables in the matrix*

$$\langle \varepsilon_{ij}^M \rangle = \varepsilon_{ij}^0 - \tfrac{1}{2} mA \langle (n_i b_j + n_j b_i) \rangle + \tfrac{1}{2} mA \langle \bar{A}_{ijkl} C_{klpq}(n_p b_q + n_q b_p) \rangle \tag{53}$$

$$\langle \sigma_{ij}^M \rangle = \sigma_{ij}^0 - \tfrac{1}{2} mA \langle \bar{D}_{ijkl}(n_k b_l + n_l b_k) \rangle. \tag{54}$$

Let us take the unit vector $\mathbf{e}_3^L$ normal to a slip plane, and choose $\mathbf{e}_1^L$ and $\mathbf{e}_2^L$ in that plane in such a manner that the new base vectors $\mathbf{e}_i^L$ relate to the fixed base vectors $\mathbf{e}_j$ by

$$\mathbf{e}_i = T_{ij}\mathbf{e}_j^L \tag{55}$$

where

$$[T_{ij}] = \begin{bmatrix} -\sin\theta & -\sin\phi\cos\theta & \cos\phi\cos\theta \\ \cos\theta & -\sin\phi\sin\theta & \cos\phi\sin\theta \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \tag{56}$$

with $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant \phi \leqslant \pi/2$.

If we assume that every dislocation loop is circular and lies on the slip plane, the components of the unit normal $\mathbf{n}(=\mathbf{e}_3^L)$ are (0, 0, 1). Then, the components of eigenstrain with respect to the local coordinate system become,

$$\begin{aligned} (\varepsilon_{11}^*)^L &= (\varepsilon_{22}^*)^L = 0 \\ (\varepsilon_{33}^*)^L &= b_3 \\ (\varepsilon_{12}^*)^L &= \tfrac{1}{2}b_1 \\ (\varepsilon_{23}^*)^L &= \tfrac{1}{2}b_2. \end{aligned} \tag{57}$$

By transforming the tensor in eqns (49)–(54) into the fixed coordinate system and then averaging the result over all orientations of slip planes, it follows that

$$\langle \tfrac{1}{2}(n_i b_j + n_j b_i)\rangle = \langle T_{i\alpha}T_{j\beta}(\varepsilon_{\alpha\beta}^*)^L\rangle \tag{58}$$

$$\langle \tfrac{1}{2}\bar{A}_{ijkl}C_{klpq}(n_p b_q + n_q b_p)\rangle = \langle T_{i\alpha}T_{j\beta}(\bar{A}_{\alpha\beta\nu\mu})^L(C_{\nu\mu\tau\sigma})^L(\varepsilon_{\tau\sigma}^*)^L\rangle \tag{59}$$

$$\langle \tfrac{1}{2}\bar{D}_{ijkl}(n_k b_l + n_l b_k)\rangle = \langle T_{i\alpha}T_{j\beta}(\bar{D}_{\alpha\beta\nu\mu})^L(\varepsilon_{\nu\mu}^*)^L\rangle \tag{60}$$

(a) *All slip planes are in parallel with the plane* $(\mathbf{e}_1, \mathbf{e}_2)$

In this case, we obtain,

$$T_{i\alpha} = \delta_{i\alpha}; \quad T_{j\beta} = \delta_{j\beta}. \tag{61}$$

By substituting eqns (58)–(60) into eqns (49)–(54), one obtains

$$\langle \varepsilon_{ij}\rangle = \varepsilon_{ij}^0 - mA(\varepsilon_{ij}^*)^L \tag{62}$$

$$\langle \sigma_{ij}\rangle = 0 \tag{63}$$

$$\langle \varepsilon_{ij}^I\rangle = \varepsilon_{ij}^0 - mA(\varepsilon_{ij}^*)^L - (1-mA)(\bar{A}_{ijkl})^L C_{klpq}(\varepsilon_{pq}^*)^L \tag{64}$$

$$\langle \sigma_{ij}^I\rangle = \sigma_{ij}^0 + (1-mA)(\bar{D}_{ijkl})^L(\varepsilon_{kl}^*)^L \tag{65}$$

$$\langle \varepsilon_{ij}^M\rangle = \varepsilon_{ij}^0 - mA(\varepsilon_{ij}^*)^L + mA(\bar{A}_{ijkl})^L C_{klpq}(\varepsilon_{pq}^*)^L \tag{66}$$

$$\langle \sigma_{ij}^M\rangle = \sigma_{ij}^0 - mA(\bar{D}_{ijkl})^L(\varepsilon_{kl}^*)^L \tag{67}$$

where $\langle \varepsilon_{ij}\rangle$ and $\langle \sigma_{ij}\rangle$ are average strain and stress, and $\langle \varepsilon_{ij}^I\rangle$ and $\langle \sigma_{ij}^I\rangle$ are the mean values of strain and stress within a dislocation loop. Here $\langle \varepsilon_{ij}^M\rangle$ and $\langle \sigma_{ij}^M\rangle$ are the mean values of strain and stress in the matrix.

(b) *The orientation of slip planes is random*

Equation (58) becomes

$$H_{ij}^1 = \langle T_{i\alpha} T_{j\beta} (\varepsilon_{\alpha\beta}^*)^L \rangle = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} T_{i\alpha} T_{j\beta} (\varepsilon_{\alpha\beta}^*)^L \cos\phi \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{68}$$

and hence

$$\begin{aligned} H_{ij}^2 &= \langle T_{i\alpha} T_{j\beta} (\bar{A}_{\alpha\beta\nu\rho})^L C_{\nu\rho\tau\sigma} (\varepsilon_{\tau\sigma}^*)^L \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} T_{i\alpha} T_{j\beta} (\bar{A}_{\alpha\beta\nu\rho})^L C_{\nu\rho\tau\sigma} (\varepsilon_{\tau\sigma}^*)^L \cos\phi \, \mathrm{d}\phi \, \mathrm{d}\theta \end{aligned} \tag{69}$$

$$H_{ij}^3 = \langle T_{i\alpha} T_{j\beta} (\bar{D}_{\alpha\beta\nu\rho})^L (\varepsilon_{\nu\rho}^*)^L \rangle = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} T_{i\alpha} T_{j\beta} (\bar{D}_{\alpha\beta\nu\rho})^L (\varepsilon_{\nu\rho}^*)^L \cos\phi \, \mathrm{d}\phi \, \mathrm{d}\theta. \tag{70}$$

By substituting eqns (68)–(70) into eqns (49)–(52), one obtains

$$\begin{aligned} \langle \varepsilon_{ij} \rangle &= \varepsilon_{ij}^0 - mAH_{ij}^1 \\ \langle \sigma_{ij} \rangle &= \sigma_{ij}^0 \end{aligned} \tag{71}$$

$$\begin{aligned} \langle \varepsilon_{ij}^I \rangle &= \varepsilon_{ij}^0 - mAH_{ij}^1 - (1 - mA) H_{ij}^2 \\ \langle \sigma_{ij}^I \rangle &= \sigma_{ij}^0 + (1 - mA) H_{ij}^3 \end{aligned} \tag{72}$$

$$\begin{aligned} \langle \varepsilon_{ij}^M \rangle &= \varepsilon_{ij}^0 - mAH_{ij}^1 + mAH_{ij}^2 \\ \langle \sigma_{ij}^M \rangle &= \sigma_{ij}^0 - mAH_{ij}^3. \end{aligned} \tag{73}$$

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APPENDIX: THE COMPONENTS OF TENSORS A, D, $\bar{A}$, $\bar{D}$

According to Kunin (1983), the tensors A and D have the symmetry of the elliposid and are defined by nine essential components. We have in the coordinate system connected to the ellipsoid axes:

$$A_{1111} = k_0[3I_{11} + (1 - 4\gamma_0) I_1] \tag{A1}$$

$$A_{1122} = k_0[I_{21} - I_1] \tag{A2}$$

$$A_{1212} = \frac{k_0}{2}[I_{21} + I_{12} + (1 - 2\gamma_0)(I_1 + I_2)] \tag{A3}$$

$$D_{1111} = r_0\left[1 - \frac{1}{8\pi}(3I_{11} + I_1)\right] \tag{A4}$$

$$D_{1122} = r_0\left\{\gamma_0 - \frac{1}{16\pi}[I_{21}+I_{12}-(1-4\gamma_0)(I_1+I_2)]\right\} \tag{A5}$$

$$D_{1212} = r_0\left\{\frac{1-\gamma_0}{2} - \frac{1}{16\pi}[I_{21}+I_{12}+(1-2\gamma_0)(I_1+I_2)]\right\} \tag{A6}$$

where

$$k_0 = \frac{1}{16\pi\mu_0(1-\gamma_0)}, \quad r_0 = \frac{2\mu_0}{1-\gamma_0}, \quad I_p = \tfrac{3}{2}v\int_0^{\infty}\frac{d\xi}{(a_p^2+\xi)\Delta(\xi)}, \quad I_{pq} = \tfrac{3}{2}va_p^2$$

$$\int_0^{\infty}\frac{d\xi}{(a_p^2+\xi)(a_q^2+\xi)\Delta(\xi)}, \quad \Delta(\xi) = \sqrt{(a_1^2+\xi)(a_2^2+\xi)(a_3^2+\xi)}; \quad p,q = 1,2,3. \tag{A7}$$

v is the mean ellipsoid volume. The remaining six tensor components are obtained by a cyclic replacement of the indices, 1, 2, 3.

The components of tensors $\bar{A}$ and $\bar{D}$ can be found as limits of tensors A and D in the local coordinate system:

$$\bar{A}_{3333} = 8\pi k_0(1-2\gamma_0); \quad \bar{A}_{1313} = 4\pi k_0(1-2\gamma_0)$$

$$\bar{A}_{2323} = 4\pi k_0(1-\gamma_0)$$

$$\bar{D}_{1111} = \bar{D}_{2222} = r_0$$

$$\bar{D}_{1122} = r_0\gamma_0; \quad \bar{D}_{1212} = \frac{r_0}{2}(1-\gamma_0)$$

which are the only non-zero components of tensor $\bar{A}$ and $\bar{D}$.